Islamic University
College of Technical Engineering
Department of Computer Technical Engineering
Academic Year
2019-2020

Subject : Engineering Analysis

Class: 3rd year

Assistant Prof.Dr.Haitham K.Ali

Numerical methods

All the techniques we have so far met for solving differential equations are known as analytical methods, and these methods give rise to a solution in terms of elementary functions such as $\sin x, e^x$, x^3, etc.

When using numerical methods it is important to note that they result in approximate solutions to differential equations and solutions are only calculated at discrete intervals of the independent variable, typical x or t. We shall begin by examining the first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

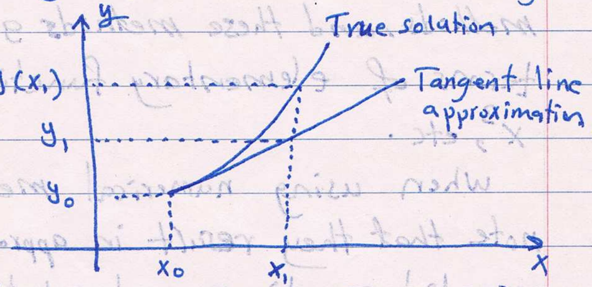
Subject to the initial condition $y(x_0) = y_0$; Usually the solution is obtained at equally spaced values of x, and we call this spacing the step size, denoted by h. By choosing a suitable value of h we can control the accuracy of the approximate solution obtained.

1. Euler's method

The first-order differential equation:

$$\frac{dy}{dx} = f(x, y)$$

the quantity $\frac{dy}{dx}$ represents the gradient of the function, we see that the differential equation tells us the gradient of the required function. Given the initial condition $y = y_0$ when $x = x_0$ we can picture this single point as shown in figure below



Moreover, we know the gradient of the solution here. Because $\frac{dy}{dx} = f(x,y)$ we see that

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f(x_0, y_0)$$

which equals the gradient of the solution at $x = x_0$. The straight line has gradient $f(x_0, y_0)$ and passes through (x_0, y_0). It can be shown that its equation is therefore

$$y - y_0 = f(x_0, y_0)(x - x_0)$$
$$y = y_0 + f(x_0, y_0)(x - x_0)$$

when $x = x_1$, the y coordinate is then given by

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

and since $x_1 - x_0 = h$, we find

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

This equation can be used to find y_1. We then regard (x_1, y_1) as known. From this known point the whole process is then repeated using the formula

③

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

and we can therefore generate a whole sequence of approximate values of y.

Ex: Use Euler's method with $h = 0.25$ to obtain a numerical solution of

$$\frac{dy}{dx} = -x y^2$$

subject to $y(0) = 2$, giving approximate values of y for $0 \leq x \leq 1$.

a) Find $y(1)$

b) determine the exact solution for comparison.

Solution

a- The corresponding x values are $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, and $x_4 = 1$. Euler's method becomes

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

$$y_{i+1} = y_i + h\,(-x_i y_i^2) \quad \text{with } x_0 = 0,\ y_0 = 2$$

we find

$x_0 = 0 \Rightarrow y_1 = y_0 - 0.25\,(x_0 y_0^2) = 2 - 0.25\,(0 \times 2^2) = 2$

$x_1 = 0.25 \Rightarrow y_2 = y_1 - 0.25\,(x_1 y_1^2) = 2 - 0.25\,(0.25 \times 2^2) = 1.75$

$x_2 = 0.5 \Rightarrow y_3 = y_2 - 0.25\,(x_2 y_2^2) = 1.75 - 0.25\,(0.5 \times (1.75)^2) = 1.367$

$x_3 = 0.75 \Rightarrow y_4 = 1.367 - 0.25\,(0.75 \times (1.367)^2) = 1.017$

$x_4 = 1 \Rightarrow y_5 = 1.017 - 0.25\,(1 \times (1.017)^2) =$

b) The exact solution can be found by separating the variables

$$\int \frac{dy}{y^2} = -\int x\,dx$$

so that

$$-\frac{1}{y} = -\frac{x^2}{2} + c$$

Imposing $y(0) = 2$ gives $c = -\frac{1}{2}$ so that

$$-\frac{1}{y} = -\frac{x^2}{2} - \frac{1}{2}$$

Finally

$$y = \frac{2}{x^2+1}$$

i	x_i	y_i numerical	$y(x_i)$ exact
0	0	2	2
1	0.25	2	1.882
2	0.5	1.75	1.6
3	0.75	1.367	1.28
4	1.0	1.017	1.00

2- Picard's Method

consider the initial value problem given by

$$\frac{dy}{dx} = f(x,y) \quad ; \qquad y(x_0) = y_0$$

$\therefore \; dy = f(x,y) \cdot dx$, Integrating both sides, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

Note: $y(x_1) = y_1$
$y(x_2) = y_2$
$\vdots$

To obtain the first approximation, replacing y by y_0 on RHS

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

Similarly

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$\vdots$$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad , \text{ where } y(x_0) = y_0$$

Remark: Picard's method can be applied only to limited types of problems, which can be integrated successively.

Ex: using picard's method, solve the initial value problem $\frac{dy}{dx} = x + y$; $y(0) = 1$, upto 3 approximations.

Solution: Given $f(x,y) = x + y$, $x_0 = 0, y_0 = 1$

using picard's approximation

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = y_0 + \int_{x_0}^{x} (x + y_0)\, dx$$

$x_0 = 0$, $y_0 = 1$

$$y_1 = 1 + \int_{0}^{x} (x+1)\, dx = 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx = 1 + \int_{0}^{x} (x + y_1)\, dx$$

$$y_2 = 1 + \int_{0}^{x} \left[x + \left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$y_2 = 1 + x + x^2 + \frac{x^3}{6}$$

3rd Approximation

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx = 1 + \int_{0}^{x} (x + y_2)\, dx$$

$$y_3 = 1 + \int_{0}^{x} \left[x + \left(1 + x + x^2 + \frac{x^3}{6}\right)\right] dx$$

$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Ex: Obtain a solution for values of x between 1 and 2 of

$$\frac{dy}{dx} = \frac{y}{x}$$

subject to $y=1$ when $x=1$ using Euler's method. Use a step size of $h=0.2$, working throughout to two decimal places of accuracy. Compare your answer with analytical solution.

<u>Solution</u> $f(x,y) = \frac{y}{x}$, $y_0 = 1$ at $x_0 = 1$ and $h = 0.2$. With a step size of 0.2 $x_1 = 1.2$, $x_2 = 1.4$, $\ldots$, $x_5 = 2$. We need to calculate $y_1, y_2, y_3, \ldots, y_5$. Euler's method $y_{i+1} = y_i + h \cdot f(x_i, y_i)$ reduces to

$$y_{i+1} = y_i + 0.2\left(\frac{y_i}{x_i}\right)$$

$$y_1 = y_0 + 0.2\left(\frac{y_0}{x_0}\right) = 1 + 0.2\left(\frac{1}{1}\right) = 1.2$$

$$y_2 = y_1 + 0.2\left(\frac{y_1}{x_1}\right) = 1.2 + 0.2 \times \frac{1.2}{1.2} = 1.4$$

$$\vdots$$

To obtain the analytical solution

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

so that

$$\ln y = \ln x + \ln D = \ln Dx$$

Therefore

$$y = Dx$$

$$1 = D \cdot 1 \implies D = 1$$

$$y = x$$

i	x_i	y_i Numerical	y_i Exact
0	1.0	1.0	1.0
1	1.2	1.2	1.2
2	1.4	1.4	1.4
3	1.6	1.6	1.6
4	1.8	1.8	1.8
5	2.0	2.0	2.0

H.W using Euler's method, estimate $y(3)$ given $y' = \frac{x+y}{x}$ $y(2) = 1$

use $h = 0.5$ and 0.25. Solve this equation analytically and compare your numerical solutions with the true solution

ans Exact = $y = x \ln|x| - 0.193x$

$y_{app}(3) = 2.6$ at $h = 0.5$ and $y_{Ex} = 2.7165$

$y_{app}(3) = 2.6561$ at $h = 0.25$ and $y_{Ex} = 2.7165$

Ex: using picard's method, obtain the solution of
$$\frac{dy}{dx} = x(1+x^3y);\ y(0)=3 \text{ at } x=0.1.$$

Solution $y_0 = 3$, $x_0 = 0$, $f(x,y) = x(1+x^3y)$

1^{st} Approximation
$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx = 3 + \int_0^x x(1+x^3y_0)\,dx$$
$$y_1 = 3 + \int_0^x x(1+3x^3)\,dx$$
$$y_1 = 3 + \frac{x^2}{2} + \frac{3x^5}{5}$$

2^{nd} Approximation
$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx$$
$$y_2 = 3 + \int_0^x x\left[1 + x^3\left(3 + \frac{x^2}{2} + \frac{3x^5}{5}\right)\right]dx$$
$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5} + \frac{x^7}{14} + \frac{3x^{10}}{50}$$

Clearly y_1 and y_2 are coincident upto 3 terms

$\therefore$ Let $y = 3 + \frac{x^2}{2} + \frac{3x^5}{5}$

Also $y(0.1) = 3 + \frac{(0.1)^2}{2} + \frac{3(0.1)^5}{5} = 3.00501$

Ex: using picard's method, solve the initial value problem $\frac{dy}{dx} = xy$; $y(1) = 2$, upto 3 approximation.

Solution: $f(x,y) = xy$, $x_0 = 1$, $y_0 = 2$

1^{st} App.: $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = 2 + \int_{1}^{x} 2x\,dx$

$$y_1 = 2 + [x^2]_1^x = 1 + x^2$$

2^{nd} appr.: $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx = 2 + \int_{1}^{x} x \cdot y_1\,dx$

$$y_2 = 2 + \int_1^x x(1+x^2)\,dx = \frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}$$

3^{rd} appr.: $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx = 2 + \int_{1}^{x} x y_2\,dx$

$$y_3 = 2 + \int_1^x x\left[\frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}\right]dx = \frac{29}{24} + \frac{5x^2}{8} + \frac{x^4}{8} + \frac{x^6}{24}$$

$$\therefore y_3 = \frac{29}{24} + \frac{5x^2}{8} + \frac{x^4}{8} + \frac{x^6}{24}$$

H.W: If $\frac{dy}{dx} = 1 + xy$ and $y(0) = 1$. calculate $y(0.1)$ and $y(0.2)$ using picard's method.

ans: $y_1 = 1 + x + \frac{x^2}{2}$ $\qquad y(0.1) = 1.10534$

$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$ $\qquad y(0.2) = 1.22286$

$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$

$y = y_3$

END OF LECTURE

ANY QUESTIONS?